CBC FREMANTLE

3CDMAS

Semester 1 Exam 2010

SECTION ONE: Resource Free

NAME:	Section Marks		Total Marks	Final Score
Marking Key	A: /40	%		
	B: /80	%	/120	%

TIME ALLOWED FOR THIS PAPER:

Reading time before commencing the paper	5 minutes
Working time for paper	50 minutes
Total Pages:	12 pages
Total Questions	10 questions
Total Marks:	40 marks

MATERIALS REQUIRED/RECOMMENDED FOR THIS SECTION

- ***To be provided by the supervisor***

  This Question/Answer Booklet
  Formula Sheet

- ***To be provided by the candidate***

  Standard items: pens, pencils, sharpener, eraser, correction fluid, ruler, highlighters
  Special items: nil

IMPORTANT NOTICE TO CANDIDATES

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

STRUCTURE OF THIS PAPER

Section	Number of questions available	Number of questions to be answered	Suggested working time (minutes)	Marks available
Section One: Calculator-free	6	6	50	40
Section Two: Calculator-assumed	13	13	100	80
			Total	120

INSTRUCTIONS TO CANDIDATES

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2. Answer the questions according to the following instructions.

   Section One: Write answers in this Question/Answer Booklet.
   All questions should be answered.

   Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked. It is recommended that you **do not use pencil** except in diagrams.

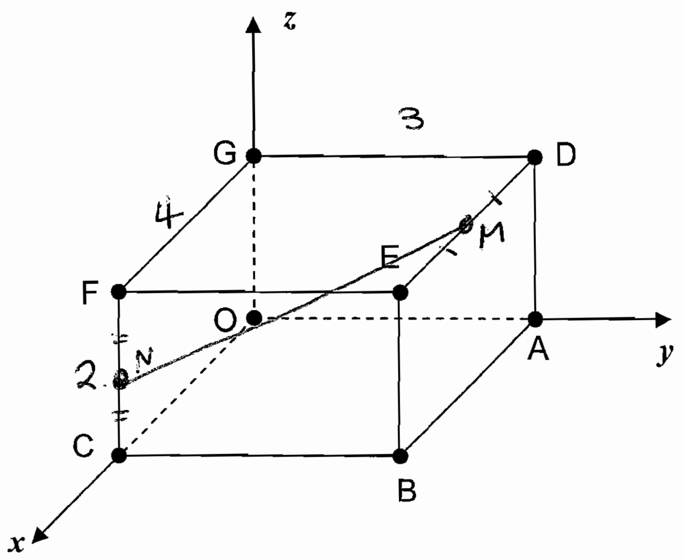
3. You must be careful to confine your responses to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

Question 1 [1 + 1 + 1 + 2 = 5 marks]

The figure below shows the cuboid OABCGDEF, with:

$\overline{OA} = 3, \overline{OC} = 4$ and $\overline{OG} = 2$.



Given that:

$$\boldsymbol{i} = \begin{pmatrix}1\\0\\0\end{pmatrix},\ \boldsymbol{j} = \begin{pmatrix}0\\1\\0\end{pmatrix} \text{ and } \boldsymbol{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

express the following in terms of $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$.

(a) $\overrightarrow{OE} =$ $4i + 3j + 2k$ ✓ [1]

(b) $\overrightarrow{DF} =$ $4i - 3j$ ✓ [1]

(c) $\overrightarrow{AF} =$ $4i - 3j + 2k$ ✓ [1]

(d) If M is the mid-point of $\overline{DE}$ and N is the mid-point of $\overline{FC}$, then find $|MN|$. [2]

$$|MN| = |\underset{\sim}{N} - \underset{\sim}{M}| = \left|\begin{pmatrix}4\\0\\1\end{pmatrix} - \begin{pmatrix}2\\3\\2\end{pmatrix}\right| ✓$$

$$= \left|\begin{matrix}-2\\-3\\-1\end{matrix}\right| = \sqrt{4+9+1} = \underline{\sqrt{14}} ✓$$

5

Question 2 [1 + 1 + 1 + 4 = 7 marks]

An object moves with constant speed, in ms[-1], and its position $\boldsymbol{r_1}$ from the origin at any time t, is given by:

$$\frac{2-x}{3} = \frac{y+1}{2} = \frac{z}{5}$$

(a) What is the velocity vector of this object? [1]

$t = \frac{2-x}{3} \rightarrow x = 2 - 3t$

$t = \frac{y+1}{2} \rightarrow y = -1 + 2t$

$t = \frac{z}{5} \rightarrow z = 0 + 5t$

$\therefore \underset{\sim}{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix}$

$\therefore \underset{\sim}{v} = \boxed{\begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix}}$ ✓

(b) What is the position vector of this object at $t = 4$? [1]

$\underset{\sim}{r}(4) = \begin{pmatrix} 2 - 3\times 4 \\ -1 + 2\times 4 \\ 0 + 5\times 4 \end{pmatrix} = \boxed{\begin{pmatrix} -10 \\ 7 \\ 20 \end{pmatrix}}$ ✓

The position vector of another object is given by: $\boldsymbol{r_2} = \begin{pmatrix} -8 \\ 29 \\ 20 \end{pmatrix} + t \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$

(c) What is the relative velocity of the second object with respect to the first? [1]

$_{r_2}V_{r_1} = V_{r_2} - V_{r_1}$

$= \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix}$

$= \boxed{\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \frac{m}{s}}$ ✓

(can also accept $\sqrt{14}$ m/s)

3

(Question 2 continued)

(d) Show that these two object collide, and find the time and position where this occurs.

[4]

$\underset{\sim}{r}_1 = \underset{\sim}{r}_2$

$$\therefore \begin{pmatrix} 2-3t \\ 2t-1 \\ 5t \end{pmatrix} = \begin{pmatrix} -8-2t \\ 29-t \\ 20+3t \end{pmatrix} \checkmark$$

$$5t = 20+3t$$
$$2t = 20$$
$$\underline{t=10} \text{ sec.} \checkmark$$

$$\therefore r_1(10) = \begin{pmatrix} 2-30 \\ 20-1 \\ 50 \end{pmatrix} = \begin{pmatrix} -28 \\ 19 \\ 50 \end{pmatrix}$$

$$r_2(10) = \begin{pmatrix} -8-20 \\ 29-10 \\ 20+30 \end{pmatrix} = \begin{pmatrix} -28 \\ 19 \\ 50 \end{pmatrix} \checkmark$$

(checking $t=10$ is correct)

$\therefore$ collide at $\begin{pmatrix} -28 \\ 19 \\ 50 \end{pmatrix}$ at $t=10$ sec.

$\checkmark$

Question 3 [4 marks]

Use De Moivre's theorem to determine the three cubic roots of $z = 4\sqrt{3} - 4i$.
Express your answers using exact values.

[4]

$z = 4\sqrt{3} - 4i$ $\quad r = \sqrt{16\times 3 - 16} = \sqrt{64} = 8$

$\theta = \tan^{-1}\left(\frac{-4}{4\sqrt{3}}\right) = \boxed{-\frac{\pi}{6}}$

$\therefore z^3 = 8 \operatorname{cis}\left(\frac{-\pi}{6}\right)$ ✓

$\Rightarrow z = \left[8 \operatorname{cis}\left(\frac{-\pi}{6}\right)\right]^{1/3}$

$= 2 \operatorname{cis}\left(\frac{-\pi}{18} + \frac{2k\pi}{3}\right)$ ✓ for $k = 0, 1, 2$.

$\therefore$ $k=0 \Rightarrow z_1 = 2 \operatorname{cis}\left(\frac{-\pi}{18}\right)$

$k=1 \Rightarrow z_2 = 2 \operatorname{cis}\left(\frac{11\pi}{18}\right)$ ✓

$k=0 \Rightarrow z_3 = 2 \operatorname{cis}\left(\frac{-13\pi}{18}\right)$ ✓

- $\frac{-1}{18} + \frac{2}{3} = \frac{-1+12}{18} = \frac{11}{18}$
- $\frac{-1}{18} + \frac{4}{3} = \frac{-1+24}{18} = \frac{23}{18} \rightarrow \frac{18+5}{18} \Rightarrow \frac{-18+5}{18} = -\frac{13}{18}$

4

Question 4 [2 + 2 + 2 = 6 marks]

A particle follows an elliptical path described by the parametric equations given below.

$$x(t) = 2\cos t \quad \text{and} \quad y(t) = \sin t$$

(a) Find the Cartesian equation of the path described by this particle. [2]

- $\cos t = \frac{x}{2} \rightarrow \cos^2 t = \frac{x^2}{4}$
- $\sin t = y \rightarrow \sin^2 t = y^2$

$\therefore \sin^2 t + \cos^2 t = y^2 + \frac{x^2}{4} = 1$

$\therefore \frac{x^2}{4} + y^2 = 1$ ✓✓

(b) Obtain $\frac{dy}{dx}$ in terms of x and y. [2]

$\frac{d}{dx}\left(\frac{x^2}{4} + y^2\right) = \frac{d}{dx}(1)$

$\therefore \frac{2x}{4} + 2y\frac{dy}{dx} = 0$ ✓

$y\frac{dy}{dx} = -\frac{x}{4}$

$\therefore \frac{dy}{dx} = \frac{-x}{4y}$ ✓

(c) Evaluated $\frac{dy}{dx}$ for $t = \frac{\pi}{4}$. [2]

$x\left(\frac{\pi}{4}\right) = 2\cos\left(\frac{\pi}{4}\right) = 2\cdot\frac{\sqrt{2}}{2} = \sqrt{2}$ ✓

$y\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\therefore \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \frac{-\sqrt{2}}{4\cdot\frac{\sqrt{2}}{2}} = \frac{-1}{2}$ ✓

6

Question 5 [7 marks]

Choose an appropriate substitution to determine:

$$\int \frac{\sqrt{4-x^2}}{3}\,dx$$ [7]

Show ALL working.

Let $x = 2\sin u$ ✓ (suitable choice)

$\frac{dx}{du} = 2\cos u$

$\therefore \; dx = 2\cos u\, du$ ✓ (differential)

$x^2 = 4\sin^2 u$

$4 - x^2 = 4 - 4\sin^2 u$

$= 4(1-\sin^2 u)$

$= 4\cos^2 u$

$\therefore \sqrt{4-x^2} = \sqrt{4\cos^2 u} = 2\cos u$ ✓

$\therefore = \int \frac{2\cos u}{3} \cdot 2\cos u\, du$

$= \frac{4}{3}\int \cos^2 u\, du$ ✓

$= \frac{4}{3}\int\left(\frac{1+\cos 2u}{2}\right)du$ ✓

$= \frac{2}{3}\int du + \frac{2}{3}\int \cos 2u\, du$

$= \frac{2u}{3} + \frac{2}{3} \times \frac{\sin 2u}{2} + C$ ✓

$= \frac{2}{3}\sin^{-1}\left(\frac{x}{2}\right) + \frac{2}{3}\sin u\cos u + C$

$= \frac{2}{3}\sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{3}\sqrt{1-\frac{x^2}{4}} + C$ ✓ (answer in terms of x)

Note: this is not essential but ideal.
can also accept $\frac{2}{3}\sin\left(2\sin^{-1}\left(\frac{x}{2}\right)\right)$.

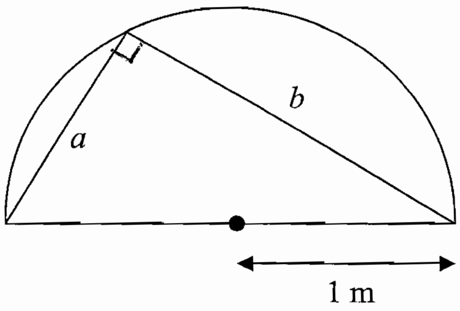
($\sin u = \frac{x}{2} \Rightarrow 1-\cos^2 u = \frac{x^2}{4} \rightarrow \cos u = \sqrt{1-\frac{x^2}{4}} \;\therefore\; \frac{1}{3}\sin 2u = \frac{2}{3}\sin u\cos u$.)

7

Question 6 [6 marks]

A triangle is inscribed within a semicircle of radius $r = 1$ metre as shown.
Use calculus to find the values of a and b that maximise the area of the triangle.
Show ALL working.

[6]



$$A = \frac{ab}{2}$$

$$a^2 + b^2 = 2^2$$

$$\therefore b = \sqrt{4-a^2}.$$

$$A(a) = \frac{a}{2}\sqrt{4-a^2}.$$

$$\frac{dA}{da} = \frac{1}{2}\sqrt{4-a^2} + \frac{a}{2} \times \frac{1}{2} \times (4-a^2)^{-1/2} \times (-2a).$$

$$= \frac{\sqrt{4-a^2}}{2} - \frac{a^2}{2\sqrt{4-a^2}} = \frac{4-a^2-a^2}{2\sqrt{4-a^2}} = \frac{4-2a^2}{2\sqrt{4-a^2}}$$

$$\therefore \frac{dA}{da} = \frac{2-a^2}{\sqrt{4-a^2}} = 0$$

$$\Rightarrow a^2 = 2 \Rightarrow a = \pm\sqrt{2}$$

$$\therefore b = \sqrt{4-a^2} = \sqrt{4-2} = \sqrt{2}$$

$$\therefore \boxed{a = b = \sqrt{2}.}$$

END OF SECTION ONE

 6

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Semester 1 Exam 2010

SECTION TWO: Resource Rich

NAME: Marking Key	Total Marks	Section Score
	/80	%

TIME ALLOWED FOR THIS PAPER:

Reading time before commencing the paper	10 minutes
Working time for paper	100 minutes
Total Pages:	20 pages
Total Questions	13 questions
Total Marks:	80 marks

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Question 1 [8 marks]

Consider the three functions given below.

$y + 1 = 3t^2$ $z = \frac{1}{1-t}$ $x + 5 = z^3$

(a) Find $\frac{dy}{dx}$ in terms of t. [5]

• $y = 3t^2 - 1 \Rightarrow \frac{dy}{dt} = 6t$ ✓

• $\frac{dz}{dt} = \frac{-(-1)}{(1-t)^2} = \frac{1}{(1-t)^2}$ ✓

• $\frac{dx}{dz} = 3z^2 = \frac{3}{(1-t)^3}$ ✓

• $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dz} \cdot \frac{dz}{dx} = 6t \cdot \frac{(1-t)^2}{1} \cdot \frac{(1-t)^2}{3} = 2t(1-t)^4$

✓ ✓

(b) Find $\frac{d^2y}{dx^2}$ in terms of t. (Do not simplify) [3]

$\frac{d^2y}{dx^2} = \frac{d}{d}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$ $\Rightarrow$ $x = \frac{1}{(1-t)^3} - 5$

$\therefore \frac{dx}{dt} = \frac{3}{(1-t)^4}$ ✓

$\therefore \frac{d^2y}{dx^2} = \left[2(1-t)^4 - 8t(1-t)^3\right] \cdot \frac{(1-t)^4}{3}$

✓ ✓

8

Question 2 [1 + 2 +2 + 2 + 3 = 10 marks]

Three position vectors are given: $A = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$ $B = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ $C = \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}$

(a) Find a unit vector in the direction of *A* with the magnitude of *B*. [1]

$|B| = \sqrt{1+1+4} = \sqrt{6}$

$|A| = \sqrt{4+9} = \sqrt{13}$

$\therefore \boxed{\underset{\sim}{u} = \frac{|B|}{|A|}\underset{\sim}{A} = \frac{\sqrt{6}}{\sqrt{3}}\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}}$ ✓

(b) Find the equation of the line L_1 that passes through *A* and *B*. [2]

$\overrightarrow{AB} = \underset{\sim}{B} - \underset{\sim}{A} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

$\therefore \underset{\sim}{r_1} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ ✓✓

(c) Find the equation of a line L_2 that passes through *C* and is perpendicular to L_1. [2]

need $\underset{\sim}{n} \perp L_1 \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 0$

$\Rightarrow$ choose $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ ✓

$\therefore \underset{\sim}{r_2} = \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ ✓

5

(Question 2 continued)

(d) Find the equation of the plane $\prod_1$: $ax + by + cz = 1$, that passes through all three points. [2]

$$\boxed{\begin{aligned} 0x + 2y + 3z &= 1 \\ x + y + 2z &= 1 \\ -x - y + 4z &= 1 \end{aligned}}$$

GC ⇒ $x = \frac{1}{3}$

$y = 0$

$x = \frac{1}{3}$

$$\therefore \boxed{\frac{x}{3} + \frac{z}{3} = 1}$$

✓✓

(e) Another plane $\prod_2$ is parallel to $\prod_1$ and is 1 unit away from $\prod_1$.
Find an equation for $\prod_2$ of the form $\boldsymbol{r} = \boldsymbol{m} + \lambda\boldsymbol{n} + \mu\boldsymbol{p}$. [3]

$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$

$|A| = \sqrt{13}$. let $\underset{\sim}{D}$ be in the direction of $\underset{\sim}{A}$
but 1 unit away from $\underset{\sim}{A}$.

$$\therefore \underset{\sim}{D} = \frac{|A| \pm 1}{|A|} \underset{\sim}{A} = \frac{\sqrt{13} \pm 1}{\sqrt{13}} \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

$$\therefore \Pi_2 : \underset{\sim}{r_2} = \frac{\sqrt{13} \pm 1}{\sqrt{13}} \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \underbrace{\lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}}_{\checkmark}$$

✓✓

 5

Question 3 [5 marks]

Find the equation of the tangent to the curve $\sin xy = \sin x + \sin y$ at the point $(0, \pi)$.

Express your answer using exact values.

[5]

$$\frac{d}{dx}\left(\sin xy\right) = \frac{d}{dx}\left(\sin x + \sin y\right)$$

$$\therefore \quad \cos xy\left(y + x\frac{dy}{dx}\right) = \cos x + \cos y\frac{dy}{dx}.$$

$$\frac{dy}{dx}\left(x\cos xy - \cos y\right) = \cos x - y\cos xy$$

$$\therefore \quad \boxed{\frac{dy}{dx} = \frac{\cos x - y\cos xy}{x\cos xy - \cos y}}$$

$$\left.\frac{dy}{dx}\right|_{(0,\pi)} = \frac{\cos 0 - \pi\cos 0}{0 - \cos\pi}$$

$$= \frac{1 - \pi}{-(-1)} = \boxed{1 - \pi}$$

$$(0, \pi) \Rightarrow \quad \therefore \quad y = (1 - \pi)x + c.$$

$$\boxed{c = \pi}$$

$$\therefore \quad \boxed{y = (1 - \pi)x + \pi}$$

5

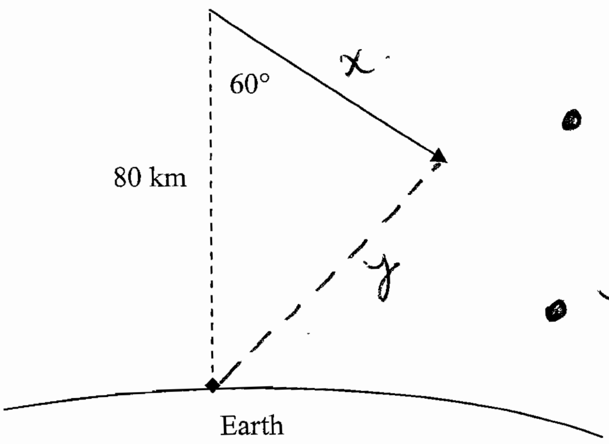
Question 4 [6 marks]

An observatory is studying the average speed at which shooting stars enter our atmosphere. A shooting star normally starts to glow from 80km above the surface of the Earth.

One shooting star enters the atmosphere at an angle of 60° with the vertical as shown. When the shooting star is 90 km from the observatory, the star is measured to be moving at 1200 kmh^{-1} from the observatory itself. (i.e. relative to the observatory).

Find the linear speed of the shooting star at this instant.

[6]



$\frac{dy}{dt} = 1200$ km/h

$y^2 = x^2 + 80^2 - 2(80)x\cos 60°$

$y^2 = x^2 + 80^2 - 80x$

$\therefore 2y\frac{dy}{dx} = 2x - 80$

$\therefore \frac{dy}{dx} = \frac{x-40}{y}$

$\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt}$

$\therefore \frac{dx}{dt} = \frac{1200y}{x-40}$

$x^2 + 80^2 - 80x - 90^2 = 0$

$\therefore x = -17.45$

or

$x = 97.45$

$\therefore \frac{dx}{dt} = \frac{1200 \times 90}{97.45 - 40}$

$= 1879.9$ kmh^{-1}

Question 5 [4 marks]

The radius of the Moon was measured to be 1 737.4 km, however the digital equipment used to obtain this measurement had an error margin of 0.5%.

Use the calculus of small changes to obtain the approximate percentage of error obtained when we estimate the surface area of the Moon.

[4]

- $A = 4\pi r^2$

  $\frac{dA}{dr} = 8\pi r$ ✓

- $\frac{\delta A}{\delta r} \approx \frac{dA}{dr}$

  $\therefore \delta A \approx \delta r \frac{dA}{dr} = \delta r \cdot 8\pi r$ ✓

- $\frac{\delta A}{A} = \frac{\delta r\, 8\pi r}{4\pi r^2} = 2\frac{\delta r}{r}$ ✓

  $= 2\,(0.5\%)$

  $= \underline{1\%}$ ✓

4

Question 6 [3 + 4 = 7 marks]

Determine:

(a) $\int \cos^3 at\ dt$ where a is a constant. [3]

$= \int \cos at\,(1 - \sin^2 at)\,dt$ ✓

$= \int \cos at\,dt - \int \cos at \sin^2 at\,dt$

$= \dfrac{\sin at}{a} - \dfrac{\sin^3 at}{3a} + C$

✓ ✓

(b) $\int \frac{x+1}{\sqrt{x-1}}\,dx$ using the substitution $u^2 = x - 1$ [4]

$x = u^2 + 1$

$\frac{dx}{du} = 2u \rightarrow dx = 2u\,du$ ✓

$= \int \frac{(u^2+2)}{u} \cdot 2u\,du$

$x + 1 = u^2 + 2$

$= 2\int (u^2+2)\,du$ ✓

$= \frac{2}{3}u^3 + 4u + C$ ✓

$= \frac{2}{3}(x-1)^{3/2} + 4(x-1)^{1/2} + C$ ✓

$\left(= \frac{1}{3}(x-1)^{1/2}(2x-10) + C. \right)$ simplified.

7

Question 7 [5 marks]

Given that $y = \frac{1+\sqrt{x}}{1-\sqrt{x}}$, show that $\frac{dy}{dx} = \frac{a}{x^b(1-\sqrt{x})^c}$, and give the values of a, b and c.

Show ALL working.

[5]

$$\frac{dy}{dx} = \frac{\frac{1}{2\sqrt{x}}(1-\sqrt{x}) + \frac{1}{2\sqrt{x}}(1+\sqrt{x})}{(1-\sqrt{x})^2}$$

$$= \frac{\frac{1}{2\sqrt{x}} - \frac{1}{2} + \frac{1}{2\sqrt{x}} + \frac{1}{2}}{(1-\sqrt{x})^2}$$

$$= \frac{\frac{1}{\sqrt{x}}}{(1-\sqrt{x})^2}$$

$$= \frac{1}{x^{1/2}(1-\sqrt{x})^2}$$

$\therefore$ $a = 1$

$b = 1/2$

$c = 2$

5

Question 8 [2 + 2 + 3 = 7 marks]

A particle starts from rest at $t = 0$, and its acceleration is given by $a = \sqrt{1+4t}$ ms^{-1}.

(a) Find an expression for the velocity of the particle in terms of t. [2]

$$GC \Rightarrow \int a(t)dt = \frac{(4t+1)^{3/2}}{6} + C \quad ✓$$

$$\therefore \quad C = \frac{-1}{6} \quad ✓$$

$$\Rightarrow \quad v(t) = \frac{(4t+1)^{3/2}}{6} - \frac{1}{6}$$

(b) Find and expression of the displacement of the particle in terms of t. [2]

$$GC \Rightarrow \int v(t)dt = \frac{(4t+1)^{5/2}}{60} - \frac{x}{6} + C \quad ✓$$

$$C = \frac{-1}{60} \quad ✓$$

$$\therefore \quad x(t) = \frac{(4t+1)^{5/2}}{60} - \frac{x}{6} - \frac{1}{60}$$

(c) Find the displacement and the acceleration of the particle when its velocity is $\frac{7}{6}$ ms^{-1}. [3]

$$v(t) = \frac{(4t+1)^{3/2}}{6} - 1 = \frac{7}{6} \overset{GC.}{\Rightarrow} t = \frac{3}{4} \quad ✓$$

$$\therefore \quad a\left(\tfrac{3}{4}\right) = \sqrt{2} \text{ ms}^{-2} \quad ✓$$

$$x\left(\tfrac{3}{4}\right) = \frac{47}{120} = 0.25 \text{ m} \quad ✓$$

7

Question 9 [2 + 2 = 4 marks]

(a) Find the acute angle between the planes $2x + 3y - 5z = 10$ and $-x + 2y + 4z = 4$. [2]

$r_1 \cdot \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} = 10$ $r_2 \cdot \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = 4$

$\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = -2 + 6 - 20 = -16$ ✓

$\left| \begin{matrix} 2 \\ 3 \\ -5 \end{matrix} \right| = \sqrt{38}$ $\therefore \theta = \cos^{-1}\left(\frac{-16}{\sqrt{38}\sqrt{21}}\right)$

$\left| \begin{matrix} -1 \\ 2 \\ 4 \end{matrix} \right| = \sqrt{21}$ $= 55.5°$ ✓

(b) Find the value of k so that the vector $\begin{pmatrix} k \\ 1+k \\ 1-k \end{pmatrix}$ belongs to the plane $r \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 5$. [2]

$\begin{pmatrix} k \\ 1+k \\ 1-k \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 5$ ✓

$\Rightarrow k + 2 + 2k - 1 + k = 5$

$4k = 4$

$k = 1$ ✓

4

Question 10 [3 + 3 = 6 marks]

A particle moves with velocity $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ kmh^{-1} and passes through $\begin{pmatrix} 10 \\ -40 \\ 40 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -20 \\ 10 \end{pmatrix}$ at 6 am and 8 am respectively.

(a) Find the values of a, b and c. [3]

Let : $(t=0) \Rightarrow r(0) = \begin{pmatrix} 10 \\ -40 \\ 40 \end{pmatrix}$

$\therefore \; r = \begin{pmatrix} 10 \\ -40 \\ 40 \end{pmatrix} + 2\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ -20 \\ 10 \end{pmatrix}$

$\Rightarrow \; a = -5$ ✓

$b = 10$ ✓

$c = -15$ ✓

(b) Find where and when the particle crosses the *x-y* plane. [3]

$\Downarrow$

$z = 0$

$\therefore \quad 40 - 15t = 0$ ✓

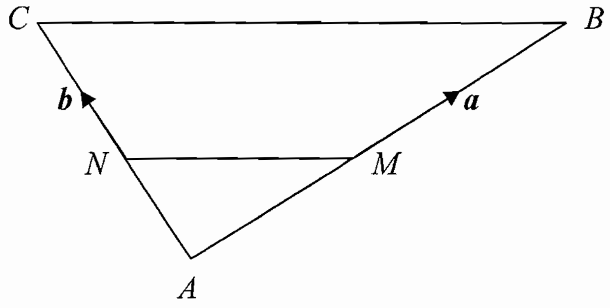
$t = \frac{8}{3}$ ✓

$\therefore \; r\left(\frac{8}{3}\right) = \begin{pmatrix} 10 - 40/3 \\ -40 + 80/3 \\ 40 - 120/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -10 \\ -40 \\ 0 \end{pmatrix}$ ✓

6

Question 11 [4 marks]

In ΔABC shown, the points M and N divide the segments $\overline{AB}$ and $\overline{AC}$ respectively in the ratio $1:3$. Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AC} = \boldsymbol{b}$.



(a) Find an expression for $\overrightarrow{BC}$ and $\overrightarrow{MN}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. [2]

- $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$ ✓
- $\overrightarrow{MN} = \frac{1}{4}(\underset{\sim}{b} - \underset{\sim}{a})$ ✓

(b) Prove that $\overrightarrow{BC} = 4\overrightarrow{MN}$ [2]

$$\overrightarrow{BC} = \underset{\sim}{b} - \underset{\sim}{a}.$$

$$= 4 \times \underbrace{\frac{1}{4}(\underset{\sim}{b} - \underset{\sim}{a})}_{\overrightarrow{MN}}$$ ✓✓

$$\therefore \overrightarrow{BC} = 4\overrightarrow{MN}$$

4

Question 12 [4 + 5 = 9 marks]

Use the fact that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0$ to find the following limits.

(a) $\lim_{x \to 0} \frac{1-\cos 2x}{x^2}$ $= \lim_{x \to 0} \frac{1 - \cos^2 x + \sin^2 x}{x^2}$ [4]

$$= \lim_{x \to 0} \frac{\sin^2 + \cos^2 - \cos^2 x + \sin^2 x}{x^2}$$

$$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2}$$

$$= 2\left(\lim_{x \to 0} \frac{\sin x}{x}\right)^2 = 2(1)^2 = 2$$

(b) $\lim_{x \to 0} \frac{2\sin x - \sin 2x}{1-\cos 2x}$ [5]

$$= \lim_{x \to 0} \frac{2\sin x - 2\sin x \cos x}{\sin^2 x + \cos^2 x - \cos^2 x + \sin^2 x}$$

$$= \lim_{x \to 0} \frac{2\sin x(1-\cos x)}{2\sin^2 x}$$

$$= \lim_{x \to 0} \frac{1-\cos x}{\sin x} \times \frac{x^{-1}}{x^{-1}}$$

$$= \lim_{x \to 0} \frac{\left(\frac{1-\cos x}{x}\right)}{\left(\frac{\sin x}{x}\right)}$$

$$= \frac{0}{1} = 0$$

9

Question 13 [3 + 2 = 5 marks]

The motion of a particle is described by the equation: $\frac{d^2x}{dt^2} + 4\pi^2 x = 0$

(a) Given that the particle begins at the origin, with positive velocity and has a maximum velocity of 8π m/sec, determine the displacement of the particle at any time *t*. [3]

SHM $\Rightarrow$ let $x(t) = A\sin(kt + \phi)$

$\therefore x(0) = A\sin(\phi) = 0$

since $A \neq 0 \Rightarrow \sin\phi = 0$

$\therefore \phi = 0$ or π

$\dot{x}(t) = kA\cos(kt + \phi)$

$\dot{x}(0) = kA\cos\phi > 0 \Rightarrow \phi = 0$

and $k > 0$

$-k^2 = -4\pi^2$

$\therefore \boxed{k = 2\pi}$

$\therefore \dot{x}(t) = 2\pi A\cos(2\pi t)$

$\dot{x}(t)_{max} = 2\pi A = 8\pi$

$\Rightarrow \boxed{A = 4}$

$\therefore \boxed{x(t) = 4\sin(2\pi t)}$

(b) Find the amplitude and the period of its motion. [2]

$\therefore A = 4$

$\boxed{T = \frac{2\pi}{n} = \frac{2\pi}{2\pi} = 1.}$

END OF PAPER

5